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MARKSCHEME

November 2013

MATHEMATICS

Higher Level

Paper 1

21 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Mark according to Scoris instructions and the document "**Mathematics HL: Guidance for e-marking November 2013**". It is essential that you read this document before you start marking. In particular, please note the following:

- Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.
- If a part is **completely correct**, (and gains all the "must be seen" marks), use the ticks with numbers to stamp full marks.
- If a part is completely wrong, stamp ***A0*** by the final answer.
- If a part gains anything else, it **must** be recorded using **all** the annotations.
- All the marks will be added and recorded by Scoris.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, for example, ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (for example, substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc*, do **not** split the marks.
- Once a correct answer to a question or part-question is seen, ignore further working.

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

*Implied marks appear in **brackets**, for example, **(M1)**, and can only be awarded if **correct** work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (**FT**) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s). To award **FT** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (for example, $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent *A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Mis-read

*If a candidate incorrectly copies information from the question, this is a mis-read (**MR**). A candidate should be penalized only once for a particular mis-read. Use the **MR** stamp to indicate that this has been a misread. Then deduct the first of the marks to be awarded, even if this is an **M** mark, but award all others so that the candidate only loses one mark.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (for example, $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

*An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. In such cases the annotation DM should be used and a brief **note** written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

Unless the question specifies otherwise, ***accept*** *equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

$$f'(x) = \left(2\cos(5x-3)\right)5 \quad \left(=10\cos(5x-3)\right)$$ ***A1***

Award ***A1*** for $\left(2\cos(5x-3)\right)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of answers

Candidates should ***NO LONGER*** *be penalized for an accuracy error* ***(AP)****.*

If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for ***FT****.*

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

No calculator is allowed. The use of any calculator on paper 1 is malpractice, and will result in no grade awarded. If you see work that suggests a candidate has used any calculator, please follow the procedures for malpractice. Examples: finding an angle, given a trig ratio of 0.4235.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

14. Candidate work

Candidates are meant to write their answers to Section A on the question paper (QP), and Section B on answer booklets. Sometimes, they need more room for Section A, and use the booklet (and often comment to this effect on the QP), or write outside the box. This work should be marked.

The instructions tell candidates not to write on Section B of the QP. Thus they may well have done some rough work here which they assume will be ignored. If they have solutions on the answer booklets, there is no need to look at the QP. However, if there are whole questions or whole part solutions missing on answer booklets, please check to make sure that they are not on the QP, and if they are, mark those whole questions or whole part solutions that have not been written on answer booklets.

SECTION A

1. $f(-2)=0$ $(\Rightarrow -24+4p-2q-2=0)$ ***M1***

$f(-1)=4$ $(\Rightarrow -3+p-q-2=4)$ ***M1***

> **Note:** In each case award the ***M*** marks if correct substitution attempted and right-hand side correct.

attempt to solve simultaneously ($2p-q=13$, $p-q=9$) ***M1***

$p=4$ ***A1***

$q=-5$ ***A1***

Total [5 marks]

2. (a) $\frac{1}{6}+\frac{1}{2}+\frac{3}{10}+a=1 \Rightarrow a=\frac{1}{30}$ ***A1***

[1 mark]

(b) $E(X)=\frac{1}{2}+2\times\frac{3}{10}+3\times\frac{1}{30}$ ***M1***

$=\frac{6}{5}$ ***A1***

> **Note:** Do not award ***FT*** marks if a is outside $[0, 1]$.

[2 marks]

(c) $\mathrm{E}(X^2)=\frac{1}{2}+2^2\times\frac{3}{10}+3^2\times\frac{1}{30}=2$ ***(A1)***

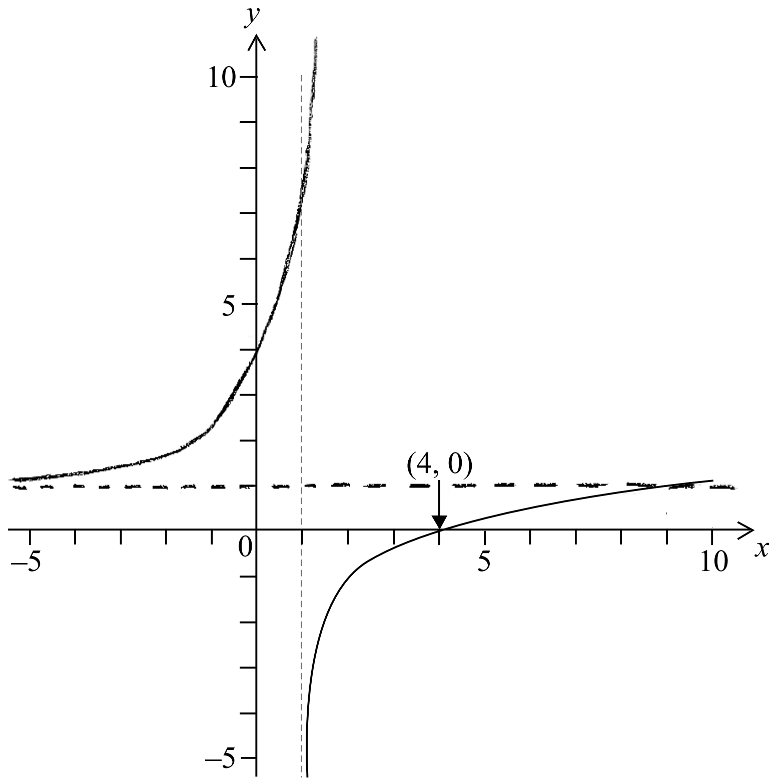
attempt to apply $\mathrm{Var}(X)=\mathrm{E}\left(X^2\right)-\left(\mathrm{E}(X)\right)^2$ ***M1***

$\left(=2-\frac{36}{25}\right)=\frac{14}{25}$ ***A1***

[3 marks]

Total [6 marks]

3. (a)



shape with y-axis intercept $(0, 4)$ *A1*

Note: Accept curve with an asymptote at $x = 1$ suggested.

correct asymptote $y = 1$ *A1*

[2 marks]

(b) range is $f^{-1}(x) > 1$ (or $]1, \infty[$) *A1*

Note: Also accept $]1, 10]$ or $]1, 10[$.

Note: Do not allow follow through from incorrect asymptote in (a).

[1 mark]

(c) $(4, 0) \Rightarrow \ln(4a + b) = 0$ *M1*

$\Rightarrow 4a + b = 1$ *A1*

asymptote at $x = 1 \Rightarrow a + b = 0$ *M1*

$\Rightarrow a = \dfrac{1}{3}, \ b = -\dfrac{1}{3}$ *A1*

[4 marks]

Total [7 marks]

4. (if $\boldsymbol{A}$ is singular then) det $\boldsymbol{A} = 0$ (seen anywhere) ***R1***

det $\boldsymbol{A} = b(b+1) - a(a+1)$ ***M1***

$b + b^2 - a - a^2 = 0$

EITHER

$b - a + (b+a)(b-a) = 0$ ***(M1)***

$(b-a)(1+b+a) = 0$ ***A1***

OR

$b - a = a^2 - b^2$ ***(M1)***

$b - a = (a+b)(a-b)$ or $-(a-b) = (a+b)(a-b)$ ***A1***

THEN

$a + b = -1$ ***A1***

Total [5 marks]

5. $3x^2y^2 + 2x^3y\dfrac{\mathrm{d}y}{\mathrm{d}x} + 3x^2 - 3y^2\dfrac{\mathrm{d}y}{\mathrm{d}x} + 9\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$ ***M1M1A1***

Note: First ***M1*** for attempt at implicit differentiation, second ***M1*** for use of product rule.

$\left(\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{3x^2y^2 + 3x^2}{3y^2 - 2x^3y - 9}\right)$

$\Rightarrow 3x^2 + 3x^2y^2 = 0$ ***(A1)***

$\Rightarrow 3x^2\left(1 + y^2\right) = 0$

$x = 0$ ***A1***

Note: Do not award ***A1*** if extra solutions given *eg* $y = \pm 1$.

substituting $x = 0$ into original equation ***(M1)***

$y^3 - 9y = 0$

$y(y+3)(y-3) = 0$

$y = 0,\ y = \pm 3$

coordinates $(0, 0), (0, 3), (0, -3)$ ***A1***

Total [7 marks]

6. $n=1$: $1^3+11=12$

$=3\times 4$ **or** a multiple of 3 ***A1***

assume the proposition is true for $n=k$ ($ie\ k^3+11k=3\mathrm{m}$) ***M1***

Note: Do not award ***M1*** for statements with "Let $n=k$".

consider $n=k+1$: $(k+1)^3+11(k+1)$ ***M1***

$=k^3+3k^2+3k+1+11k+11$ ***A1***

$=k^3+11k+(3k^2+3k+12)$ ***M1***

$=3(m+k^2+k+4)$ ***A1***

Note: Accept $k^3+11k+3(k^2+k+4)$ or statement that $k^3+11k+(3k^2+3k+12)$ is a multiple of 3.

true for $n=1$, and $n=k$ true $\Rightarrow n=k+1$ true

hence true for all $n\in\mathbb{Z}^+$ ***R1***

Note: Only award the final ***R1*** if at least 4 of the previous marks have been achieved.

Total [7 marks]

7. (a) **METHOD 1**

$a+ar=10$ ***A1***

$a+ar+ar^2+ar^3=30$ ***A1***

$a+ar=10\Rightarrow ar^2+ar^3=10r^2$ **or** $ar^2+ar^3=20$ ***M1***

$10+10r^2=30$ **or** $r^2(a+ar)=20$ ***A1***

$\Rightarrow r^2=2$ ***AG***

METHOD 2

$\frac{a\left(1-r^2\right)}{1-r}=10$ and $\frac{a\left(1-r^4\right)}{1-r}=30$ ***M1A1***

$\Rightarrow\frac{1-r^4}{1-r^2}=3$ ***M1***

leading to either $1+r^2=3$ (or $r^4-3r^2+2=0$) ***A1***

$\Rightarrow r^2=2$ ***AG***

[4 marks]

continued

Question 7 continued

(b) (i) $a + a\sqrt{2} = 10$

$\Rightarrow a = \dfrac{10}{1+\sqrt{2}}$ **or** $a = 10\left(\sqrt{2}-1\right)$ ***A1***

(ii) $S_{10} = \dfrac{10}{1+\sqrt{2}}\left(\dfrac{\sqrt{2}^{10}-1}{\sqrt{2}-1}\right)\left(=10\times 31\right)$ ***M1***

$= 310$ ***A1***

[3 marks]

Total [7 marks]

8. (a) $\sin(x+y)\sin(x-y)$

$= (\sin x\cos y + \cos x\sin y)(\sin x\cos y - \cos x\sin y)$ ***M1A1***

$= \sin^2 x\cos^2 y + \sin x\sin y\cos x\cos y - \sin x\sin y\cos x\cos y - \cos^2 x\sin^2 y$

$= \sin^2 x\cos^2 y - \cos^2 x\sin^2 y$ ***A1***

$= \sin^2 x\left(1-\sin^2 y\right) - \sin^2 y\left(1-\sin^2 x\right)$ ***A1***

$= \sin^2 x - \sin^2 x\sin^2 y - \sin^2 y + \sin^2 x\sin^2 y$

$= \sin^2 x - \sin^2 y$ ***AG***

[4 marks]

(b) $f(x) = \sin^2 x - \dfrac{1}{4}$

range is $f \in \left[-\dfrac{1}{4}, \dfrac{3}{4}\right]$ ***A1A1***

Note: Award ***A1*** for each end point. Condone incorrect brackets.

[2 marks]

(c) $g(x) = \dfrac{1}{\sin^2 x - \dfrac{1}{4}}$

range is $g \in \left]-\infty, -4\right] \cup \left[\dfrac{4}{3}, \infty\right[$ ***A1A1***

Note: Award ***A1*** for each part of range. Condone incorrect brackets.

[2 marks]

Total [8 marks]

9. (a) $\log_2(x-2) = \log_4\left(x^2-6x+12\right)$

EITHER

$\log_2(x-2) = \dfrac{\log_2\left(x^2-6x+12\right)}{\log_2 4}$ ***M1***

$2\log_2(x-2) = \log_2\left(x^2-6x+12\right)$

OR

$\dfrac{\log_4(x-2)}{\log_4 2} = \log_4\left(x^2-6x+12\right)$ ***M1***

$2\log_4(x-2) = \log_4\left(x^2-6x+12\right)$

THEN

$(x-2)^2 = x^2-6x+12$ ***A1***

$x^2-4x+4 = x^2-6x+12$

$x = 4$ ***A1*** ***N1***

[3 marks]

(b) $x^{\ln x} = \mathrm{e}^{(\ln x)^3}$

taking ln of both sides or writing $x = \mathrm{e}^{\ln x}$ ***M1***

$(\ln x)^2 = (\ln x)^3$ ***A1***

$(\ln x)^2(\ln x - 1) = 0$ ***(A1)***

$x = 1,\ x = \mathrm{e}$ ***A1A1*** ***N2***

Note: Award second ***(A1)*** only if factorisation seen or if two correct solutions are seen.

[5 marks]

Total [8 marks]

SECTION B

10. (a) (i) $f'(x) = e^{-x} - xe^{-x}$ ***M1A1***

(ii) $f'(x) = 0 \Rightarrow x = 1$

coordinates $\left(1, e^{-1}\right)$ ***A1***

[3 marks]

(b) $f''(x) = -e^{-x} - e^{-x} + xe^{-x} \left(= -e^{-x}(2-x)\right)$ ***A1***

substituting $x = 1$ into $f''(x)$ ***M1***

$f''(1)\left(= -e^{-1}\right) < 0$ hence maximum ***R1AG***

[3 marks]

(c) $f''(x) = 0 \quad (\Rightarrow x = 2)$ ***M1***

coordinates $\left(2, 2e^{-2}\right)$ ***A1***

[2 marks]

(d) (i) $g(x) = \dfrac{x}{2} e^{-\frac{x}{2}}$ ***A1***

(ii) coordinates of maximum $\left(2, e^{-1}\right)$ ***A1***

(iii) equating $f(x) = g(x)$ and attempting to solve $xe^{-x} = \dfrac{x}{2} e^{-\frac{x}{2}}$

$\Rightarrow x\left(2e^{\frac{x}{2}} - e^{x}\right) = 0$ ***(A1)***

$\Rightarrow x = 0$ ***A1***

or $2e^{\frac{x}{2}} = e^{x}$

$\Rightarrow e^{\frac{x}{2}} = 2$

$\Rightarrow x = 2\ln 2 \qquad (\ln 4)$ ***A1***

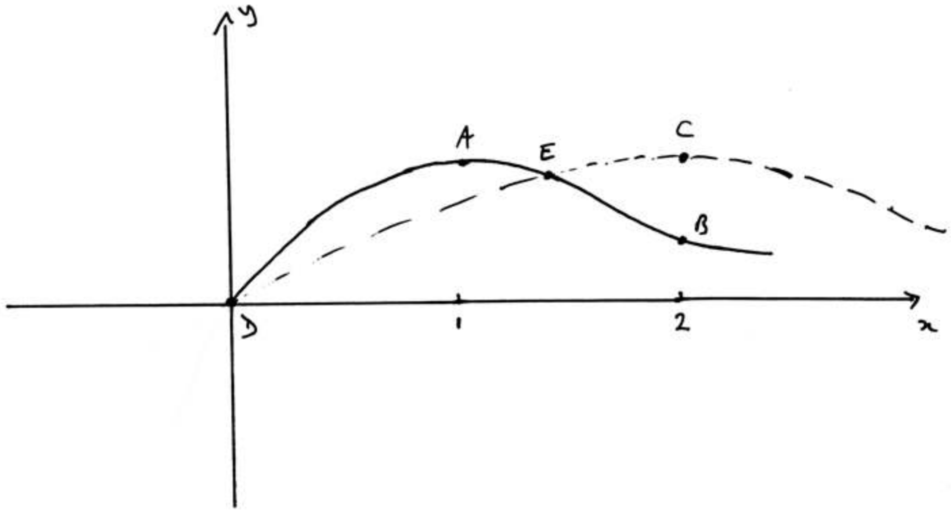
[5 marks]

Note: Award first ***(A1)*** only if factorisation seen or if two correct solutions are seen.

continued …

Question 10 continued

(e)



A4

> **Note:** Award ***A1*** for shape of f, including domain extending beyond $x = 2$.
> Ignore any graph shown for $x < 0$.
> Award ***A1*** for A and B correctly identified.
> Award ***A1*** for shape of g, including domain extending beyond $x = 2$.
> Ignore any graph shown for $x < 0$. Allow follow through from f.
> Award ***A1*** for C, D and E correctly identified (D and E are interchangeable).

[4 marks]

(f) $A = \int_0^1 \frac{x}{2}\mathrm{e}^{-\frac{x}{2}}\,\mathrm{d}x$ ***M1***

$= \left[-x\mathrm{e}^{-\frac{x}{2}}\right]_0^1 - \int_0^1 -\mathrm{e}^{-\frac{x}{2}}\,\mathrm{d}x$ ***A1***

> **Note:** Condone absence of limits or incorrect limits.

$= -\mathrm{e}^{-\frac{1}{2}} - \left[2\mathrm{e}^{-\frac{x}{2}}\right]_0^1$

$= -\mathrm{e}^{-\frac{1}{2}} - \left(2\mathrm{e}^{-\frac{1}{2}} - 2\right) = 2 - 3\mathrm{e}^{-\frac{1}{2}}$ ***A1***

[3 marks]

Total [20 marks]

11. (a) $\overrightarrow{\mathrm{CA}} = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$ ***(A1)***

$\overrightarrow{\mathrm{CB}} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ ***(A1)***

> **Note:** If $\overrightarrow{\mathrm{AC}}$ and $\overrightarrow{\mathrm{BC}}$ found correctly award ***(A1) (A0)***.

$\overrightarrow{\mathrm{CA}} \times \overrightarrow{\mathrm{CB}} = \begin{vmatrix} i & j & k \\ 1 & -2 & -1 \\ 2 & 0 & 1 \end{vmatrix}$ ***(M1)***

$\begin{pmatrix} -2 \\ -3 \\ 4 \end{pmatrix}$ ***A1***

[4 marks]

(b) **METHOD 1**

$\frac{1}{2}\left|\overrightarrow{\mathrm{CA}} \times \overrightarrow{\mathrm{CB}}\right|$ ***(M1)***

$= \frac{1}{2}\sqrt{(-2)^2 + (-3)^2 + 4^2}$ ***(A1)***

$= \frac{\sqrt{29}}{2}$ ***A1***

METHOD 2

attempt to apply $\frac{1}{2}|\mathrm{CA}||\mathrm{CB}|\sin C$ ***(M1)***

$\mathrm{CA.CB} = \sqrt{5}.\sqrt{6}\cos C \Rightarrow \cos C = \frac{1}{\sqrt{30}} \Rightarrow \sin C = \frac{\sqrt{29}}{\sqrt{30}}$ ***(A1)***

area $= \frac{\sqrt{29}}{2}$ ***A1***

[3 marks]

continued …

Question 11 continued

(c) **METHOD 1**

$$\mathbf{r}.\begin{pmatrix}-2\\-3\\4\end{pmatrix}=\begin{pmatrix}1\\0\\0\end{pmatrix}.\begin{pmatrix}-2\\-3\\4\end{pmatrix}$$ ***M1A1***

$\Rightarrow -2x-3y+4z=-2$ ***A1***

$\Rightarrow 2x+3y-4z=2$ ***AG***

METHOD 2

$-2x-3y+4z=\mathrm{d}$

substituting a point in the plane ***M1A1***

$\mathrm{d}=-2$ ***A1***

$\Rightarrow -2x-3y+4z=-2$

$\Rightarrow 2x+3y-4z=2$ ***AG***

Note: Accept verification that all 3 vertices of the triangle lie on the given plane.

[3 marks]

continued ...

Question 11 continued

(d) **METHOD 1**

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & -4 \\ 4 & -1 & -1 \end{vmatrix} = \begin{pmatrix} -7 \\ -14 \\ -14 \end{pmatrix}$$ ***M1A1***

$$\mathbf{n} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$

$z = 0 \Rightarrow y = 0,\ x = 1$ ***(M1)(A1)***

$$L_1 : \mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$ ***A1***

Note: Do not award the final ***A1*** if $\mathbf{r} =$ is not seen.

METHOD 2

eliminate 1 of the variables, *eg x* ***M1***
$-7y + 7z = 0$ ***(A1)***
introduce a parameter ***M1***
$\Rightarrow z = \lambda,$

$y = \lambda,\ x = 1 + \dfrac{\lambda}{2}$ ***(A1)***

$$\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$ or equivalent ***A1***

Note: Do not award the final ***A1*** if $\mathbf{r} =$ is not seen.

METHOD 3

$z = t$ ***M1***
write *x* and *y* in terms of $t \Rightarrow 4x - y = 4 + t,\ 2x + 3y = 2 + 4t$ or equivalent ***A1***
attempt to eliminate *x* or *y* ***M1***
x, y , z expressed in parameters
$\Rightarrow z = t,$

$y = t,\ x = 1 + \dfrac{t}{2}$ ***A1***

$$\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$$ or equivalent ***A1***

Note: Do not award the final ***A1*** if $\mathbf{r} =$ is not seen.

[5 marks]

continued …

Question 11 continued

(e) **METHOD 1**

direction of the line is perpendicular to the normal of the plane

$$\begin{pmatrix}16\\ \alpha\\ -3\end{pmatrix}.\begin{pmatrix}1\\ 2\\ 2\end{pmatrix}=0$$ ***M1A1***

$16+2\alpha-6=0 \Rightarrow \alpha=-5$ ***A1***

METHOD 2

solving line/plane simultaneously

$16(1+\lambda)+2\alpha\lambda-6\lambda=\beta$ ***M1A1***

$16+(10+2\alpha)\lambda=\beta$

$\Rightarrow \alpha=-5$ ***A1***

METHOD 3

$$\begin{vmatrix}2 & 3 & -4\\ 4 & -1 & -1\\ 16 & \alpha & -3\end{vmatrix}=0$$ ***M1***

$2(3+\alpha)-3(-12+16)-4(4\alpha+16)=0$ ***A1***

$\Rightarrow \alpha=-5$ ***A1***

METHOD 4

attempt to use row reduction on augmented matrix ***M1***

to obtain $\left(\begin{array}{ccc|c}2 & 3 & -4 & 2\\ 0 & -1 & 1 & 0\\ 0 & 0 & \alpha+5 & \beta-16\end{array}\right)$ ***A1***

$\Rightarrow \alpha=-5$ ***A1***

[3 marks]

(f) $\alpha=-5$ ***A1***

$\beta\neq 16$ ***A1***

[2 marks]

Total [20 marks]

12. (a) $z^n + z^{-n} = \cos n\theta + i\sin n\theta + \cos(-n\theta) + i\sin(-n\theta)$ ***M1***

$= \cos n\theta + \cos n\theta + i\sin n\theta - i\sin n\theta$ ***A1***

$= 2\cos n\theta$ ***AG***

[2 marks]

(b) $\left(z+z^{-1}\right)^4 = z^4 + 4z^3\left(\frac{1}{z}\right) + 6z^2\left(\frac{1}{z^2}\right) + 4z\left(\frac{1}{z^3}\right) + \frac{1}{z^4}$ ***A1***

Note: Accept $\left(z+z^{-1}\right)^4 = 16\cos^4\theta$.

[1 mark]

(c) **METHOD 1**

$\left(z+z^{-1}\right)^4 = \left(z^4 + \frac{1}{z^4}\right) + 4\left(z^2 + \frac{1}{z^2}\right) + 6$ ***M1***

$(2\cos\theta)^4 = 2\cos 4\theta + 8\cos 2\theta + 6$ ***A1A1***

Note: Award ***A1*** for RHS, ***A1*** for LHS independent of the ***M1***.

$\cos^4\theta = \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$ ***A1***

$\left(\text{or } p = \frac{1}{8},\ q = \frac{1}{2},\ r = \frac{3}{8}\right)$

METHOD 2

$\cos^4\theta = \left(\frac{\cos 2\theta + 1}{2}\right)^2$ ***M1***

$= \frac{1}{4}(\cos^2 2\theta + 2\cos 2\theta + 1)$ ***A1***

$= \frac{1}{4}\left(\frac{\cos 4\theta + 1}{2} + 2\cos 2\theta + 1\right)$ ***A1***

$\cos^4\theta = \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$ ***A1***

$\left(\text{or } p = \frac{1}{8},\ q = \frac{1}{2},\ r = \frac{3}{8}\right)$

[4 marks]

continued …

Question 12 continued

(d) $\left(z+z^{-1}\right)^6 = z^6 + 6z^5\left(\frac{1}{z}\right) + 15z^4\left(\frac{1}{z^2}\right) + 20z^3\left(\frac{1}{z^3}\right) + 15z^2\left(\frac{1}{z^4}\right) + 6z\left(\frac{1}{z^5}\right) + \frac{1}{z^6}$ ***M1***

$\left(z+z^{-1}\right)^6 = \left(z^6 + \frac{1}{z^6}\right) + 6\left(z^4 + \frac{1}{z^4}\right) + 15\left(z^2 + \frac{1}{z^2}\right) + 20$

$(2\cos\theta)^6 = 2\cos 6\theta + 12\cos 4\theta + 30\cos 2\theta + 20$ ***A1A1***

Note: Award ***A1*** for RHS, ***A1*** for LHS, independent of the ***M1***.

$\cos^6\theta = \frac{1}{32}\cos 6\theta + \frac{3}{16}\cos 4\theta + \frac{15}{32}\cos 2\theta + \frac{5}{16}$ ***AG***

Note: Accept a purely trigonometric solution as for (c).

[3 marks]

(e) $\int_0^{\frac{\pi}{2}} \cos^6\theta\,\mathrm{d}\theta = \int_0^{\frac{\pi}{2}} \left(\frac{1}{32}\cos 6\theta + \frac{3}{16}\cos 4\theta + \frac{15}{32}\cos 2\theta + \frac{5}{16}\right)d\theta$

$= \left[\frac{1}{192}\sin 6\theta + \frac{3}{64}\sin 4\theta + \frac{15}{64}\sin 2\theta + \frac{5}{16}\theta\right]_0^{\frac{\pi}{2}}$ ***M1A1***

$= \frac{5\pi}{32}$ ***A1***

[3 marks]

(f) $\mathrm{V} = \pi\int_0^{\frac{\pi}{2}} \sin^2 x\cos^4 x\,\mathrm{d}x$ ***M1***

$= \pi\int_0^{\frac{\pi}{2}} \cos^4 x\,\mathrm{d}x - \pi\int_0^{\frac{\pi}{2}} \cos^6 x\,\mathrm{d}x$ ***M1***

$\int_0^{\frac{\pi}{2}} \cos^4 x\,\mathrm{d}x = \frac{3\pi}{16}$ ***A1***

$\mathrm{V} = \frac{3\pi^2}{16} - \frac{5\pi^2}{32} = \frac{\pi^2}{32}$ ***A1***

Note: Follow through from an incorrect *r* in (c) provided the final answer is positive.

[4 marks]

continued …

Question 12 continued

(g) (i) constant term $= \binom{2k}{k} = \frac{(2k)!}{k!k!} = \frac{(2k)!}{(k!)^2}$ (accept C_k^{2k}) ***A1***

(ii) $2^{2k}\int_0^{\frac{\pi}{2}} \cos^{2k}\theta \mathrm{d}\theta = \frac{(2k)!}{(k!)^2}\frac{\pi}{2}$ ***A1***

$\int_0^{\frac{\pi}{2}} \cos^{2k}\theta \mathrm{d}\theta = \frac{(2k)!\pi}{2^{2k+1}(k!)^2}$ $\left(\text{or } \frac{\binom{2k}{k}\pi}{2^{2k+1}}\right)$ ***A1***

[3 marks]

Total [20 marks]